Razvijanje f-je u red sinusa ili kosinusa u intervalu $[0, l]$

Neka je data f-ja $f(x)$ na intervalu $[0, l]$, $l > 0$. Tada možemo definisati novu f-ju $F(x)$ na intervalu $[-l, l]$ koja se podudara sa f-jom $f(x)$ na $[0, l]$ i koja je na $[-l, l]$ parna ili neparna. F-ju $F(x)$ zovemo tada parno ili neparno produženje f-je $f(x)$.

Parno produženje f-je $f(x)$:

$$\overline{F}(x) = \begin{cases} f(x), & x \in [0, l] \\ f(-x), & x \in [0, l) \end{cases}$$

Znamo od ranije da će tada biti $b_n = 0$.

Neparno produženje f-je $f(x)$:

$$\overline{F}(x) = \begin{cases} f(x), & x \in [0, l] \\ -f(-x), & x \in [-l, 0) \end{cases}$$

Tada je $a_n = 0$.

Primjer: Data je f-ja $f(x) = \cos x$, za $x \in [0, \pi]$. Uz pomoć ove f-je napraviti f-ju $\overline{F}(x)$ koja je neparna na $[-\pi, \pi]$.

$$\overline{F}(x) = \begin{cases} \cos x, & x \in [0, \pi] \\ -\cos x, & x \in [-\pi, 0) \end{cases}$$

$\cos(-x) = \cos x$

Data je f-ja $f(x) = x^3$ za $x \in [0, 1]$. Uz pomoć ove f-je napraviti f-ju $\overline{F}(x)$ koja je parna na $[-1, 1]$

$$F(x) = \begin{cases} x^3, & \text{za } x \in [0, 1] \\ -x^3, & \text{za } x \in [-\pi, 0) \end{cases}$$

$(-x)^3 = -x^3$

Napraviti neparno produženje f-je $f(x) = x^2$ na $[-1, 1]$.

$$\overline{F}(x) = \begin{cases} x^2, & x \in [0, 1] \\ -x^2, & x \in [-\pi, 0) \end{cases}$$

$(-x)^2 = x^2$

Razviti f-ju $f(x)=x\left(\frac{\pi}{2}-x\right)$ po sinusima višestrukih uglova u intervalu $\left(0,\frac{\pi}{2}\right)$.

Rj: Furijer-ov red f-je $f(x)$ na intervalu $[a,b]$ je oblika

$$f(x)\sim \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$$

gdje se Furijeovi koeficijenti računaju po formuli

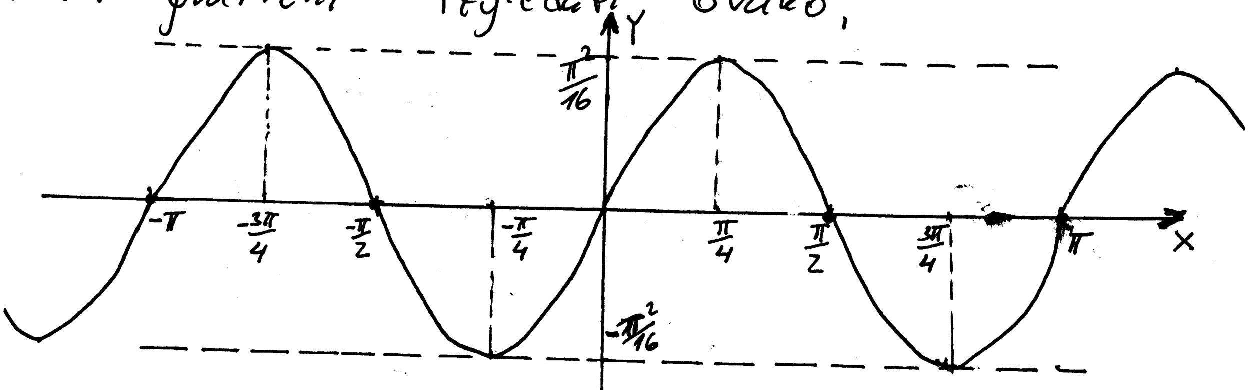
$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx.$$

Da bi razvili f-ju po sinusima višestrukih uglova trebamo naštimati da je $a_n=0$, a prema formuli za a_n, a_n će biti jednak nuli akko je interval $[a,b]$ simetričan u odnosu na 0 i ako je $f(x)$ neparna f-ja.

Prema tome da bi našu f-ju $f(x)=x\left(\frac{\pi}{2}-x\right)$ razvili u red po sinusima, pravimo neparno produženje f-je $f(x)$ (novu f-ju ćemo obilježiti sa $f^*(x)$)

$$f^*(x)=\begin{cases} f(x), & x\in\left(0,\frac{\pi}{2}\right)\\ -f(-x), & x\in\left(-\frac{\pi}{2},0\right)\end{cases}=\begin{cases} x\left(\frac{\pi}{2}-x\right), & x\in\left(0,\pi/2\right)\\ x\left(\frac{\pi}{2}+x\right), & x\in\left(-\frac{\pi}{2},0\right).\end{cases}$$

Primjetimo da f-ja $f^*(x)$ koju pretvaramo u Furijerov red će u stvari grafički izgledati ovako:



Izračunajmo Furijeove koeficijente. Posmatramo interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$b-a=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi, \quad \frac{2}{b-a}=\frac{2}{\pi}, \quad \frac{2n\pi x}{b-a}=2nx$$

$$b_n=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \underbrace{\underset{\substack{\uparrow\\ \text{neparna}\\ \text{f-ja}}}{f^*(x)}\,\underset{\substack{\uparrow\\ \text{neparna}\\ \text{f-ja}}}{\sin 2nx}}_{\text{parna f-ja}}\,dx=\frac{4}{\pi}\int_0^{\pi/2} f^*(x)\sin \sin 2nx\,dx=\frac{4}{\pi}\int_0^{\pi/2} x\left(\frac{\pi}{2}-x\right)\sin 2nx\,dx$$

$$=\frac{4}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2}x-x^2\right)\sin 2nx\,dx=\frac{4}{\pi}\cdot\frac{\pi}{2}\int_0^{\pi/2} x\sin 2nx\,dx-\frac{4}{\pi}\int_0^{\pi/2} x^2\sin 2nx\,dx \overset{(*)}{=}$$

$$I_1=2\int_0^{\pi/2} x\sin 2nx\,dx=\left|\begin{matrix} u=x & dv=\sin 2nx \\ du=dx & v=\frac{-1}{2n}\cos 2nx\end{matrix}\right|=-\frac{2x}{2n}\cos 2nx\Big|_0^{\frac{\pi}{2}}+2\cdot\frac{1}{2n}\int_0^{\pi/2}\cos 2nx\,dx$$

$$=\left(-\frac{\pi}{2n}\cos\pi n x-0\right)+\frac{1}{n}\cdot\frac{1}{2n}\sin 2nx\Big|_0^{\frac{\pi}{2}}=\frac{(-1)^{n+1}\pi}{2n}$$

$$I_2=\frac{4}{\pi}\int_0^{\pi/2} x^2\sin 2nx\,dx=\left|\begin{matrix} u=x^2 & dv=\sin 2nx\,dx \\ du=2x\,dx & v=\frac{-1}{2n}\cos 2nx\,dx\end{matrix}\right|=\frac{4}{\pi}\cdot\frac{(-1)}{2n}x^2\cos 2nx\Big|_0^{\pi/2}$$

$$+\frac{1}{2n}\cdot 2\cdot\frac{4}{\pi}\int_0^{\pi/2} x\cos 2nx\,dx=\left|\begin{matrix} u=x & dv=\cos 2nx\,dx \\ du=dx & v=\frac{1}{2n}\sin 2nx\end{matrix}\right|=\frac{-2}{n\pi}\left(\overset{\frac{\pi^2}{4}}{\left(\frac{\pi}{2}\right)^2}(-1)^n-0\right)$$

$$+\frac{4}{n\pi}\left[\underbrace{\frac{1}{2n}x\sin 2nx\Big|_0^{\frac{\pi}{2}}}_{=0}-\frac{1}{2n}\int_0^{\pi/2}\sin 2nx\right]=\frac{(-1)^{n+1}\pi}{2n}+\frac{4}{n\pi}\cdot\frac{1}{4n^2}\cos 2n\pi\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{(-1)^{n+1}\pi}{2n}+\frac{1}{n^3\pi}\left((-1)^n-1\right)$$

$$\overset{(*)}{=}\frac{(-1)^{n+1}\pi}{2n}-\frac{(-1)^{n+1}\pi}{2n}-\frac{(-1)^n-1}{n^3\pi}=\frac{1-(-1)^n}{n^3\pi}$$ vrijednost koeficijenta b_n

$$f(x)\sim\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3\pi}\sin 2nx=\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2(2k-1)x}{(2k-1)^3}$$ traženi Furijerov red

za $x\in\left(0,\frac{\pi}{2}\right)$.

(#) Razviti u intervalu $(0,\pi)$ po sinusima višestrukih lukova f-ju $f(x)=\frac{\pi}{4}$. Dobijeni razvoj upotrebite za sumiranje redova brojeva

a) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$; b) $1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\dots$;

c) $1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\dots$.

Rj. Pravimo neparno produženje f-je $f(x)$, $\overline{f(x)}=\begin{cases}\frac{\pi}{4}, & x\in(0,\pi)\\ -\frac{\pi}{4}, & x\in(-\pi,0)\end{cases}$

$\overline{f(x)}$ neparna, $a_n=0$, $n=0,1,2,\dots$

$$b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^\pi\frac{\pi}{4}\sin nx\,dx=\frac{2}{\pi}\cdot\frac{\pi}{4}\int_0^\pi\sin nx\,dx=\frac{1}{2}\cdot\left(-\frac{1}{n}\right)\cos nx\Big|_0^\pi$$

$$=-\frac{1}{2n}(\cos n\pi-\cos 0^\circ)=-\frac{1}{2n}((-1)^n-1),$$

za $n=2k$, $b_n=0$, $k=1,2,\dots$

za $n=2k+1$, $b_n=\frac{1}{n}$, $k=0,1,2,\dots$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx$$

$$\overline{f(x)}=\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}\quad\text{tj.}\quad\frac{\pi}{4}=\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1},\quad x\in(0,\pi)$$

a)

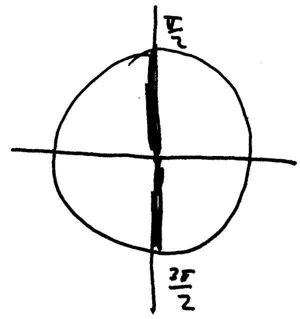
za $x=\frac{\pi}{2}$ imamo $\frac{\pi}{4}=\sum_{k=0}^{\infty}\frac{\sin\frac{(2k+1)\pi}{2}}{2k+1}$,

$k=0$, $2k+1=1$, $\sin\frac{\pi}{2}=1$

$k=1$, $2k+1=3$, $\sin\frac{3\pi}{2}=-1$

$k=2$, $2k+1=5$, $\sin\frac{5\pi}{2}=1$

⋮



$$\sin\frac{(2k+1)\pi}{2}=\begin{cases}1, & k=0,2,4,\dots\\ -1, & k=1,3,5,\dots\end{cases}$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots=\frac{\pi}{4}$$

$$\sin\frac{(2k+1)\pi}{3}=\begin{cases}\frac{\sqrt{3}}{2}, & k=0,3,6,\dots\\ 0, & k=1,4,7,\dots\\ -\frac{\sqrt{3}}{2}, & k=2,5,8,\dots\end{cases}$$

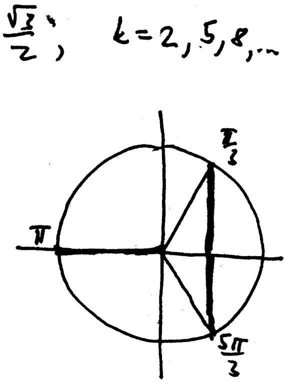
b) za $x=\frac{\pi}{3}$ imamo $\frac{\pi}{3}=\sum_{k=0}^{\infty}\frac{\sin\frac{(2k+1)\pi}{3}}{2k+1}$

$k=0$, $2k+1=1$, $\sin\frac{\pi}{3}$

$k=1$, $2k+1=3$, $\sin\pi$

$k=2$, $2k+1=5$, $\sin\frac{5\pi}{3}$

$k=3$, $2k+1=7$, $\sin\frac{7\pi}{3}$



$$\frac{\sqrt{3}}{2}\left(1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\dots\right)=\frac{\pi}{4}$$

$$\frac{\pi}{2\sqrt{3}}=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\dots$$

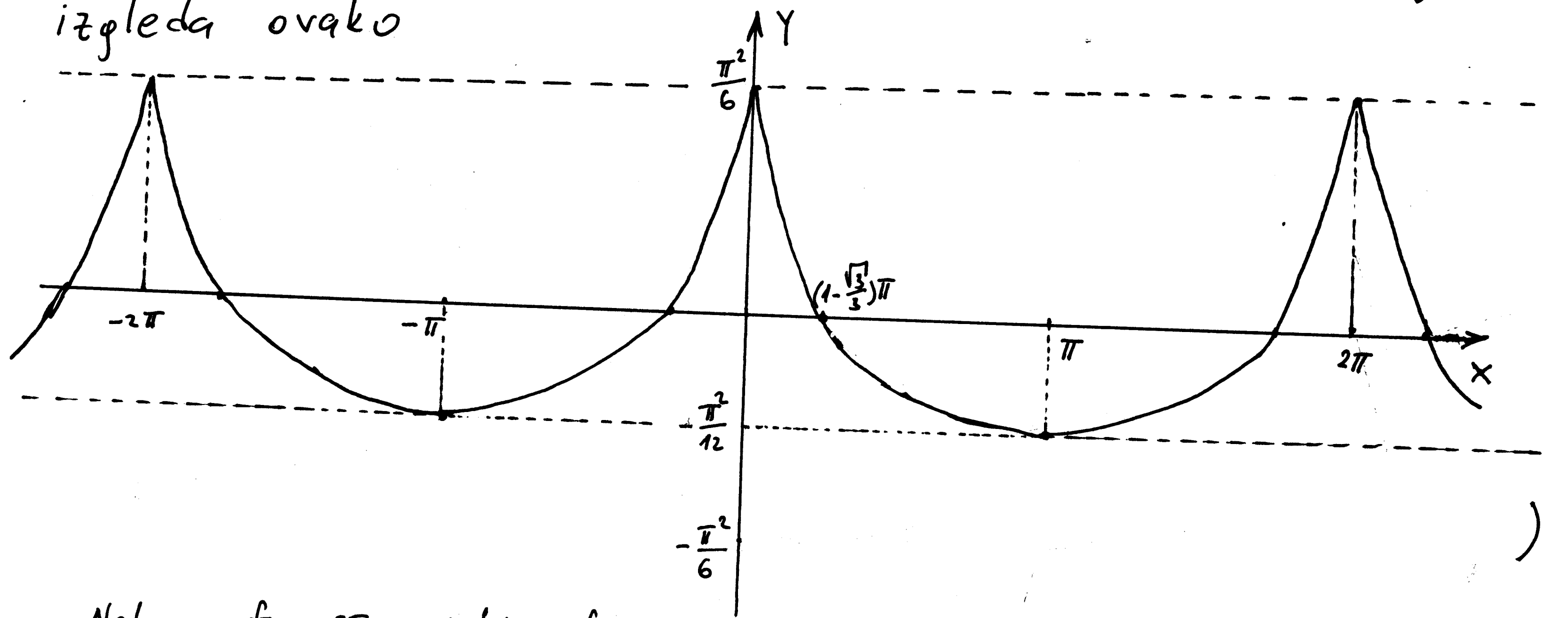
c) za vježbu

uputa: za x se uzme $\frac{\pi}{4}$

rješenje: $\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\dots$

Razviti f-ju $f(x) = \dfrac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0,\pi)$.

Rj. F-ju koju razvijamo u Furijeov red po kosinusima grafički izgleda ovako



Neka je $f(x)$ 2π periodična f-ja.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{Furijeov red f-je } f(x)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{Furijeovi koeficijenti f-je } f(x)$$

Ako je $f(x)$ parna tada je $f(x)\sin nx$ neparna $\Rightarrow$ $b_n = 0$ $\forall n \in \mathbb{Z}$

Ako je $f(x)$ neparna tada je $f(x)\cos nx$ neparna $\Rightarrow$ $a_n = 0$ $\forall n \in \mathbb{Z}$

Trebamo napraviti parno produženje f-je $f(x)$ (novu f-ju nazovimo $f^*(x)$)

$$f^*(x) = \begin{cases} f(x), & x \in (0,\pi) \\ f(-x), & x \in (-\pi,0) \end{cases} = \begin{cases} \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2), & x \in (0,\pi) \\ \frac{1}{12}(3x^2 + 6\pi x + 2\pi), & x \in (-\pi,0) \end{cases}$$

Izračunajmo Fourijeove koeficijente

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\,dx \xrightarrow{f^* \text{ parna}} \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\,dx = \frac{1}{6\pi}\left(3\cdot\frac{1}{3}x^3\Big|_0^{\pi} - \right.$$

$$\left. - 6\pi\cdot\frac{1}{2}x^2\Big|_0^{\pi} + 2\pi^2\cdot x\Big|_0^{\pi}\right) = \frac{1}{6\pi}(\pi^3 - 3\pi^3 + 2\pi^3) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\cos nx\,dx \xrightarrow{f^*(x) \text{ parna}} \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\cos nx\,dx =$$

$$= \frac{1}{6\pi}\left(3\int_0^{\pi} x^2\cos nx\,dx - 6\pi\int_0^{\pi} x\cos nx\,dx + 2\pi^2\int_0^{\pi}\cos nx\,dx\right) \overset{(*)}{=}$$

$$I_1 = \int_0^{\pi} x^2\cos nx\,dx = \left|\begin{matrix} u = x^2 & dv = \cos nx\,dx \\ du = 2x\,dx & v = \frac{1}{n}\sin nx\end{matrix}\right| = \frac{1}{n}x^2\sin nx\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx =$$

$$= \left|\begin{matrix} u = x & dv = \sin nx\,dx \\ du = dx & v = -\frac{1}{n}\cos nx\end{matrix}\right| = \frac{1}{n}\underbrace{(\pi^2\sin n\pi - 0)}_{=0} - \frac{2}{n}\left(-\frac{1}{n}x\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$

$$= \frac{2}{n^2}(\pi\cos n\pi - 0) - \frac{2}{n^2}\sin nx\Big|_0^{\pi} = (-1)^n\frac{2\pi}{n^2}$$

$$I_2 = \int_0^{\pi} x\cos nx\,dx = \left|\begin{matrix} u = x & dv = \cos nx\,dx \\ du = dx & v = \frac{1}{n}\sin nx\end{matrix}\right| = \frac{1}{n}x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx =$$

$$= \frac{1}{n}\underbrace{(\pi\sin n\pi - 0)}_{=0} - \frac{1}{n}\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi} = \frac{1}{n^2}(\cos n\pi - \cos 0) = \frac{1}{n^2}((-1)^n - 1)$$

$$I_3 = \int_0^{\pi}\cos nx\,dx = \frac{1}{n}\sin nx\Big|_0^{\pi} = \frac{1}{n}(\sin n\pi - \sin 0) = 0$$

$$\overset{(*)}{=} \frac{1}{\cancel{2\pi}}(-1)^n\frac{\cancel{2\pi}}{n^2} - \frac{1}{n^2}((-1)^n - 1) = \frac{1}{n^2}((-1)^n - (-1)^n + 1) = \frac{1}{n^2}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx, \quad x \in (0, \pi)$$

razvoj f-je $f(x)$ u red po kosinusima

(Primjetimo da dobijeni rezultat možemo iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Naime ako stavimo $x=0$ imamo $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$).

(#) F-ju $f(x)=\sin x$ razložiti u red po kosinusima u intervalu $[0,\pi]$.

Rj. Pravimo parno produženje f-je $f(x)$

$$\bar{f}(x)=\begin{cases}\sin x, & x\in[0,\pi]\\ -\sin x, & x\in[-\pi,0)\end{cases}$$

$\bar{f}(x)$ je parna f-je na $[-\pi,\pi] \Rightarrow b_n=0 \quad (n=1,2,\dots)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=-\frac{2}{\pi}\cos x\Big|_0^{\pi}=-\frac{2}{\pi}(\cos\pi-\cos 0)=-\frac{2}{\pi}\cdot(-2)=\frac{4}{\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx=\frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}\left(\sin(x+nx)+\sin(x-nx)\right)dx$$

$$\sin\alpha\cos\beta=\frac{1}{2}\sin(\alpha-\beta)+\frac{1}{2}\sin(\alpha+\beta)$$
$$\cos\alpha\cos\beta=\frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)$$
$$\sin\alpha\sin\beta=\frac{1}{2}\cos(\alpha-\beta)-\frac{1}{2}\cos(\alpha+\beta)$$

$$=\frac{1}{\pi}\int_0^{\pi}\sin(n+1)x\,dx+\frac{1}{\pi}\int_0^{\pi}\sin(1-n)x\,dx=-\frac{1}{\pi}\cdot\frac{1}{n+1}\cos(n+1)x\Big|_0^{\pi}-$$

$$-\frac{1}{\pi}\cdot\frac{1}{1-n}\cos(1-n)x\Big|_0^{\pi}=-\frac{1}{\pi(n+1)}\Big(\underbrace{\cos(n+1)\pi}_{(-1)^{n+1}}-\underbrace{\cos 0^\circ}_{1}\Big)+\frac{1}{\pi(n-1)}\Big(\underbrace{\cos(1-n)\pi}_{\cos(n-1)\pi}-\cos 0^\circ\Big)=$$

$$=\frac{1}{\pi}\left[-\frac{1}{n+1}\left((-1)^{n+1}-1\right)+\frac{1}{n-1}\left((-1)^{n-1}-1\right)\right]=\frac{1}{\pi}\left((-1)^{n+1}-1\right)\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

$$=\frac{(-1)^{n+1}-1}{\pi}\cdot\frac{\cancel{n}+1-\cancel{n}+1}{(n-1)(n+1)}=\frac{2\left[(-1)^{n+1}-1\right]}{(n^2-1)\pi},$$

$n=2k+1,\ (-1)^{n+1}-1=0$

$n=2k,\ (-1)^{n+1}-1=-2$

$$a_{2k}=\frac{-4}{\pi(4k^2-1)},\quad k=1,2,\dots$$

$$\sin x=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1},\quad x\in[-\pi,\pi]$$

razlaganje f-je $f(x)$ po kosinusima

Zadaci za vježbu

U zadacima 4377—4390 razviti u Furije-ov red date funkcije u datim intervalima.

4377. Funkciju $y=x^2$ u intervalu $(0, \pi)$ u sinusni red.

4378. Funkciju x^3 u intervalu $(-\pi, \pi)$.

4379. Funkciju $f(x)=\begin{cases} 1 & \text{za } -\pi<x<0, \\ 3 & \text{za } \quad 0<x<\pi. \end{cases}$

4380. Funkciju $f(x)=\begin{cases} 1 & \text{za } 0<x<h \\ 0 & \text{za } h<x<\pi \end{cases}$ u kosinusni red $(0<h<\pi)$.

4381. Neprekidnu funkciju $f(x)=\begin{cases} 1 & \text{za } x=0, \\ 0 & \text{za } 2h<x<\pi, \end{cases}$ i linearnu u intervalu $(0, 2h)$, — u kosinusni red. $\left(0<h<\dfrac{\pi}{2}\right)$.

4382. Funkciju $|x|$ u intervalu $(-l, l)$.

4383. Funkciju e^x-1 u intervalu $(0, 2\pi)$.

4384. Funkciju e^x u intervalu $(-l, l)$.

4385. Funkciju $\cos ax$ u intervalu $(-\pi, \pi)$ (a je neceo broj).

4386. Funkciju $\sin ax$ u intervalu $(-\pi, \pi)$ (a je neceo broj).

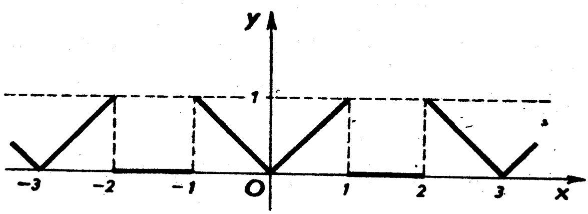
4387. Funkciju $\sin ax$ (a je ceo broj) u intervalu $(0, \pi)$ u kosinusni red.

4388. Funkciju $\cos ax$ (a je ceo broj) u intervalu $(0, \pi)$ u sinusni red.

4389. Funkciju $\operatorname{sh} ax$ u intervalu $(-\pi, \pi)$.

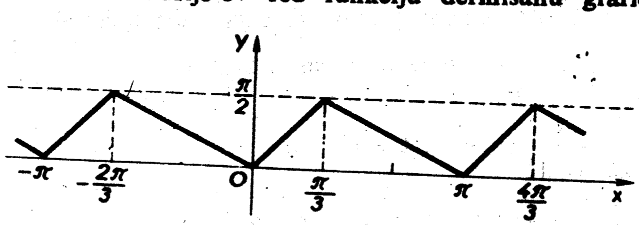
4390. Funkciju $\operatorname{ch} x$ u intervalu $(0, \pi)$ u kosinusni red i u sinusni red.

4391. Razviti u Furije-ov red funkciju definisanu grafikom na sl. 74.



Sl. 74

4392*. Razviti u Furije-ov red funkciju definisanu grafički na sl. 75.



Sl. 75

Rješenja

4377. $\dfrac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left\{\dfrac{\pi^2}{n}+\right.$

$\left.+\dfrac{2}{n^3}\left[(-1)^n-1\right]\right\}\sin nx.$

4378. $\sum_{n=1}^{\infty}(-1)^n\left(\dfrac{12}{n^3}-\dfrac{2\pi^2}{n}\right)\sin nx.$

4379. $2+\dfrac{4}{\pi}\sum_{n=0}^{\infty}\dfrac{\sin(2n+1)x}{2n+1}.$

4380. $\dfrac{2h}{\pi}\left[\dfrac{1}{2}+\sum_{n=1}^{\infty}\dfrac{\sin nh}{nh}\cos nx\right].$

4381. $\dfrac{2h}{\pi}\left[\dfrac{1}{2}+\sum_{n=1}^{\infty}\left(\dfrac{\sin nh}{nh}\right)^2\cos nx\right].$

4382. $\dfrac{l}{2}-\dfrac{4l}{\pi^2}\sum_{n=0}^{\infty}\dfrac{\cos\left[\dfrac{(2n+1)\pi x}{l}\right]}{(2n+1)^2}.$

4383. $\dfrac{e^{2\pi}-1}{\pi}\left[\dfrac{1}{2}+\sum_{n=1}^{\infty}\left(\dfrac{\cos nx}{1+n^2}-\right.\right.$

$\left.\left.-\dfrac{n\sin x}{1+n^2}\right)\right]-1.$

4384. $\dfrac{e^l-e^{-l}}{2l}+l(e^l-e^{-l})\sum_{n=1}^{\infty}\dfrac{(-1)^n\cos\dfrac{n\pi x}{l}}{l^2+n^2\pi^2}+$

$+\pi(e^l-e^{-l})\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}n\sin\dfrac{n\pi x}{l}}{l^2+n^2\pi^2}=$

$=\operatorname{sh}l\left[\dfrac{1}{l}+2\sum_{n=1}^{\infty}(-1)^n\dfrac{l\operatorname{sos}\dfrac{n\pi x}{l}-\pi n\sin\dfrac{n\pi x}{l}}{l^2+n^2\pi^2}\right].$

4385. $\dfrac{2\sin\pi a}{\pi}\left(\dfrac{1}{2a}+\dfrac{a\cos x}{1-a^2}-\dfrac{a\cos 2x}{2^2-a^2}+\cdots\right).$

4386. $\dfrac{2\sin\pi a}{\pi}\left(\dfrac{\sin x}{1-a^2}-\dfrac{2\sin 2x}{2^2-a^2}+\dfrac{3\sin 3x}{3^2-a^2}-\cdots\right).$

4387. $\sin ax=\begin{cases}\dfrac{4a}{\pi}\left[\dfrac{\cos x}{a^2-1}+\dfrac{\cos 3x}{a^2-3^2}+\dfrac{\cos 5x}{a^2-5^2}+\cdots\right] & \text{ako je } a \text{ paran broj.}\\[2ex] \dfrac{4a}{\pi}\left[\dfrac{1}{2a^2}+\dfrac{\cos 2x}{a^2-2^2}+\dfrac{\cos 4x}{a^2-4^2}+\cdots\right] & \text{ako je } a \text{ neparan broj.}\end{cases}$

4388. $\cos ax=\begin{cases}-\dfrac{4}{\pi}\left[\dfrac{\sin x}{a^2-1^2}+\dfrac{3\sin 3x}{a^2-3^2}+\dfrac{5\sin 5x}{a^2-5^2}+\cdots\right] & \text{ako je } a \text{ paran broj.}\\[2ex] -\dfrac{4}{\pi}\left[\dfrac{2\sin 2x}{a^2-2^2}+\dfrac{4\sin 4x}{a^2-4^2}+\dfrac{6\sin 6x}{a^2-6^2}+\cdots\right] & \text{ako je } a \text{ neparan broj.}\end{cases}$

4389. $\dfrac{2\operatorname{sh}a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{a^2+n^2}\sin nx.$

4390. $\dfrac{\operatorname{sh}\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}(-1)^n\dfrac{\cos nx}{1+n^2}\right];\quad \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{1-(-1)^n\operatorname{ch}\pi}{1+n^2}n\sin nx.$

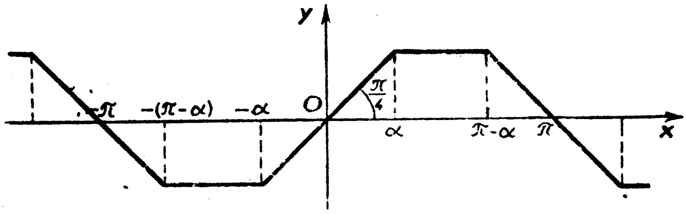
4391. $f(x)=\dfrac{1}{3}+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{\sin\dfrac{2\pi n}{3}}{n}-\right.$

$\left.-\dfrac{3\left(1-\cos\dfrac{2\pi n}{3}\right)}{2\pi n^2}\right]\cos\dfrac{2\pi nx}{3}=$

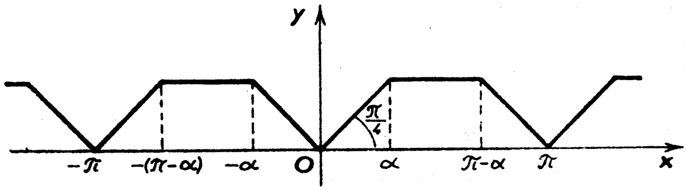
$=\dfrac{1}{3}+\dfrac{\sqrt{3}}{\pi}\left(\dfrac{\cos\dfrac{2\pi x}{3}}{1}-\dfrac{\cos\dfrac{4\pi x}{3}}{2}+\dfrac{\cos\dfrac{8\pi x}{3}}{4}-\cdots\right)-$

$-\dfrac{9}{2\pi^2}\left(\dfrac{\cos\dfrac{2\pi x}{3}}{1^2}+\dfrac{\cos\dfrac{4\pi x}{3}}{2^2}+\dfrac{\cos\dfrac{8\pi x}{3}}{4^2}+\cdots\right).$

4393*. Razviti u Furije-ov red funkcije čiji su grafici prikazani na sl. 76 i 77.



Sl. 76



Sl. 77

4394. Razviti funkciju $x(\pi-x)$ u sinusni red u intervalu $(0, \pi)$, i koristeći dobijeni rezultat naći zbir reda

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots+\frac{(-1)^{n-1}}{(2n-1)^3}+\cdots$$

4395. Data je funkcija $\varphi(x)=(\pi^2-x^2)^2$.

a) Uveriti se da ova funkcija zadovoljava jednakosti:

$$\varphi(-\pi)=\varphi(\pi),\ \varphi'(-\pi)=\varphi'(\pi)\ \text{ i }\ \varphi''(-\pi)=\varphi''(\pi)$$

$$[\text{ali } \varphi'''(-\pi)\neq\varphi'''(\varphi)].$$

b) Koristeći dobijene jednakosti razviti funkciju $\varphi(x)$ u Furije-ov red u intervalu $(-\pi, \pi)$.

c) Izračunati zbir reda

$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots+\frac{(-1)^{n-1}}{n^4}+\cdots$$

Rješenja

4392*. $$f(x)=\frac{\pi}{6}+\frac{3}{2\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{3}}{n^2}\left(\cos\frac{n\pi}{3}\sin 2nx-\sin\frac{nx}{3}\cos 2nx\right)=$$

$$=\frac{\pi}{6}+\frac{3\sqrt{3}}{8\pi}\left(\frac{\sin 2x}{1^2}-\frac{\sin 4x}{2^2}+\frac{\sin 8x}{4^2}-\frac{\sin 10x}{5^2}+\cdots\right)-$$

$$-\frac{9}{8\pi}\left(\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 8x}{4^2}+\frac{\cos 10x}{5^2}+\cdots\right).$$

Iskoristiti rezultat zadatka 4368.

4393*. 1) $$f(x)=\frac{4}{\pi}\left(\frac{\sin\alpha\cdot\sin x}{1^2}+\frac{\sin 3\alpha\cdot\sin 3x}{3^2}+\cdots\right)$$

2) $$f(x)=\frac{\alpha(\pi-\alpha)}{\pi}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos 2n\alpha}{n^2}\cos 2nx=$$

$$=\frac{\alpha(\pi-\alpha)}{\pi}-\frac{2}{\pi}\left(\frac{\sin^2\alpha\cdot\cos 2x}{1^2}+\frac{\sin^2 2\alpha\cdot\cos 4x}{2^2}+\cdots\right).$$

Iskoristiti rezultat zadatka 4371.

4394. $$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3};\ \frac{\pi^3}{32}.$$

4395. $$\frac{8}{15}\pi^4-48\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^4};\ \text{c) } \frac{7}{720}\pi^4.$$